

Beyond Lebesgue and Baire III: Steinhaus' Theorem and its descendants

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Abstract

We generalize from the real line to normed groups (i.e., the invariantly metrizable, right topological groups) a result on infinite combinatorics, valid for ‘large’ subsets in both the category and measure senses, which implies both Steinhaus’s Theorem and many of its descendants. We deduce the inherent measure-category duality in this setting directly from properties of analytic sets. We apply the result to extend the Pettis Theorem and the Continuous Homomorphism Theorem to normed groups, i.e., beyond metrizable topological groups, and in a companion paper deduce automatic joint continuity results for group operations.

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1 Introduction

Our theme is motivated by the classical analysis of regularly varying functions – developed originally on the real line in the study of Tauberian theorems in the 1930s. Recognized as a key tool by Feller [F] in probability, through its connection with limit theorems (asymptotics) and domains of attraction, the concept found an ever widening circle of applications (analytic number theory, complex analysis, differential equations – see [BGT], the definitive account of the classical theory). Its continued key role in modern applications to stochastic processes and the more recent extreme-value theories in the context of function spaces prompted a reappraisal – primarily topological.

In the classical theory the basic tool was the Steinhaus (and its category relative, the Pettis-Picard) sum-theorem. However, the recent redevelopment, providing generalization (beyond the line) and unification (of the underlying measure-category duality), as alluded to in the title, equips natural group actions with a topology and so brings it into contiguity with Effros

Open Mapping-like principles – small wonder that the approach has also yielded a non-separable Effros Theorem ([Ost-E]).

The key here is a modest “generic” local property of a metrizable group, much weaker than the ‘global’ notions of genericity that go back to Truss and Lascar (cf. [Tr-92]), or ‘ample’ genericity introduced by Shelah in [HHLS], more recently studied as ‘mutual’ genericity ([Tr]) – for more on this approach to genericity refer to the end of this section.

Expressed informally, the property of interest here asserts that for any fixed target set and fixed convergent sequence in some underlying space, with limit a distinguished point, “generically all” members in an “admissible locality” of the group (all but negligibly many in the Baire category sense) will embed a subsequence into the target set. These embeddings, or “shifts”, might be translations when the underlying space is the group itself; more generally they may be members of a group acting on the underlying space, e.g. its self-homeomorphisms, or inner-automorphisms (in the case of topological groups). In view of the passage to a subsequence, it is natural to call this a *shift-compactness* of the underlying space (under a specified group of shifts), borrowing a probabilistic term from [Par]. Van Mill [vM-09] considers, under the term ‘separation property’, the closely related ‘shift’ property requiring to shift *arbitrary* countable sets into a comeagre target (describing this as a “pushing apart”).

For the property to hold one needs some form of topological completeness in the target set. The paper’s main contribution is to verify this property (once it is defined properly) suitably generally – in regard to both the group and the target set, and to draw out several consequences but relying on less than topological completeness. We adopt much the same context as van Mill in his recent generalization [vM-04] of the Effros Mapping Theorem; as van Mill points out, the group G need carry only a unilaterally invariant metric – here right-invariant for definiteness and denoted d_R^G – and be analytic, though not necessarily separable, as we shall demonstrate. We take the target set T also analytic, and for simplicity also in G . Analyticity provides just enough of a substitute for topological completeness and is a natural assumption on account of the preservation under continuity of analyticity and so of the Baire property. The latter feature is crucial in applications such as Effros’ Theorem. It is convenient to follow a norm language, defined by $\|x\| := d_R^G(x, e_G)$, see [BOst-N] and compare the review by Pestov [Pe].

The main deductions are that such a group is (i) a Baire space, (ii) satisfies a “squared” Steinhaus-Pettis Theorem (or to “exponent 2”, borrowing

the recent useage from an allied area – cf. [RoSo]), (iii) supports an automatic continuity theorem for Baire (= Baire property) homomorphisms (cf. [HJ], [Ro]). When the group is locally compact the results are dualized explicitly to a corresponding “Haar-measure” context (though this could also be derived using a Haar-density topology, following [Mil-Ost]). In a companion paper [Ost-Joint], we deduce an automatic joint-continuity results for group operations.

Since the group need only be right-topological this prompts the question as to when it is in fact a topological group; some answers are deduced.

The paper is arranged as follows. We recall *normed groups* in Section 2.1 (connecting them with the Birkhoff-Kakutani metrization theorem, here viewed as in fact a normability result); the notion of *analyticity* together with a Cantor-like theorem appropriate to analytic spaces in Section 2.2; Section 2.3 closes by indicating a natural generalization of analyticity in the *non-separable case* (for further recent developments see [Ost-AB]). The main result of Section 3 is the Analytic Shift-theorem, asserting in the particular case of a topological group that if a target set T is analytic and non-meagre then, for any sequence $z_n \rightarrow e_G$ converging to the identity, the shifted sequence tz_n is in T for quasi all $t \in T$. (In the more general normed setting this is achieved with ‘trans-conjugacies’). Section 4 is concerned with locally-compact normed groups, where the measure analogue of the shift theorem is as simple as the category one for *topological groups*; this section exploits the existence of an invariant Haar measure and a corresponding displacement lemma (for which see Wilczyński and Kharazishvili [WKh]), which is of course a step away from the Weil topology.

We close here with a brief indication of similarities to the model-theoretically motivated strand of genericity mentioned earlier, which is limited to Polish spaces (completely metrizable and separable). Kechris and Rosendal [KR], in studying orbit structure, are concerned also with shifting countable sets into their own complements, a similar property to the one due to van Mill (see above). However, they do so using ‘generic’ elements, distinguished by having co-meagre orbits under group action. Accordingly, they must make the strong assumption that such special elements exist. (But then these exhaust almost all of the space, and the ‘generically everything is generic’ arguments are simpler.) In fact, working in an even stronger environment, they exploit Shelah’s more convenient notion of ample genericity involving n -tuples (for n arbitrary) of mutually generic elements possessing comeagre orbits; these orbits are obtained from a ‘diagonal’ action (i.e. a group

member acts identically on each component of the n -tuple). This context is convenient for constructing small shifts to move some generic element into a non-meagre target set (albeit not necessarily having the Baire property), whilst fixing other prescribed elements. A critical tool there is the Fubini-like Kuratowski-Ulam Theorem (which unfortunately fails in a non-separable metric context, as shown in [P], cf. [vMP], but see [FNR]). Our local notion has been exploited in many contexts, e.g. in the study of subadditivity – for a survey see [Ost-S] (for n -ary shifts with n fixed see the ‘consecutive embeddings’ of [BOst-Funct, Th. 3.1] and [BOst-KCC, Th. 3.6]; for older work see literature cited in [BOst-H]). See [BBM] for a recent localization of the ‘ample’ notion via a refining group-norm topology (as defined in §2).

2 Preliminaries: analyticity and norms

We begin by recalling the definition of normed groups, a category of spaces broader than the metrizable topological groups and then the notion of analyticity, which is found here to be a good replacement for completeness. Our interest in analyticity is motivated by two facts: naturally occurring subgroups of a topological group may be Borel but fail to be \mathcal{G}_δ , so are not topologically complete (see [FaSol] for a study of Borel subgroups of Polish groups); nevertheless, a sharpened form of the Effros Theorem, due to van Mill, in which the group action is supplied by an analytic group with topology derived from a norm, includes these in its remit. We wish to verify that the shift-compactness property also holds for them. Furthermore, we are free to use the fact that an analytic set has the Baire property (cf. [Kech, Th. 21.6], the Lusin-Sierpiński Theorem, and the closely related, Nikodym Theorem [Kech, Cor. 29.14], cf. the treatment in [Kur-1, Cor. 1 p. 482] or [Jay-Rog, pp. 42-43], and in [Ost-AB] in the non-separable case). We refer to this result as the *Lusin-Sierpiński-Nikodym Theorem*, abbreviated to LSN.

2.1 Normed groups

Definitions. 1. For T an algebraic group with neutral element e , say that $\|\cdot\| : T \rightarrow \mathbb{R}_+$ is a *group-norm* ([BOst-N]) if the following properties hold:

- (i) *Subadditivity* (Triangle inequality): $\|st\| \leq \|s\| + \|t\|$;
- (ii) *Positivity*: $\|t\| > 0$ for $t \neq e$ and $\|e\| = 0$;
- (iii) *Inversion* (Symmetry): $\|t^{-1}\| = \|t\|$.

Then $(T, \|\cdot\|)$ is called a *normed-group*.

2. The group-norm generates a right and a left *norm topology* via the right-invariant and left-invariant metrics $d_R^T(s, t) := \|st^{-1}\|$ and $d_L^T(s, t) := \|s^{-1}t\| = d_R^T(s^{-1}, t^{-1})$. In the right norm-topology the right shift $\rho_y(s) := sy$ is a uniformly continuous homeomorphism, since $d_R(sy, ty) = d_R(s, t)$; likewise in the left norm-topology the left shift $\lambda_x(t) = xt$ is a uniformly continuous homeomorphism. Since $d_L^T(t, e) = d_L^T(e, t^{-1}) = d_R^T(e, t)$, convergence at e is identical under either topology. In the absence of a qualifier, the ‘right’ norm topology is to be understood.

3. Note that under the right norm topology $B(x, r) = \{t : d_R(t, x) < r\} = B(e_T, r)x$.

4. Also of interest is the *symmetrization metric* $d_S^T := \max\{d_R^T, d_L^T\}$, studied in the companion paper [Ost-Joint] and in [Ost-AB].

The Birkhoff-Kakutani Theorem ([HR] Th. 8.3) asserts that a metrizable topological group has an equivalent right-invariant metric. For naturally occurring examples of normed groups see the companion paper [Ost-Joint, §4]. In fact, a close inspection of Kakutani’s metrizability proof for topological groups yields the following characterization of normed groups; we indicate the minor adjustments needed to deduce it.

Birkhoff-Kakutani Normability Theorem. *A first-countable right topological group X is a normed group iff inversion and multiplication are continuous at the identity.*

Proof. A normed group under its right-norm topology is a first-countable right topological group. It follows directly from the defining three axioms that inversion and multiplication are continuous at the identity e .

For the converse, consider any open neighbourhood U of e . Continuity of multiplication at e implies that there is an open neighbourhood V of e such that $V^4 \subseteq U$. Next, using continuity of inversion at e , one may choose an open neighbourhood $N \subseteq V$ of e such that $N^{-1} \subseteq V$. Then, as right shifts are homeomorphisms, $W := NN^{-1} = \bigcup\{Ng^{-1} : g \in N\}$ is an open neighbourhood of e with $W^{-1} = W \subseteq V^2$. So, since $W^2 \subseteq V^4 \subseteq U$, we conclude that for any open neighbourhood U of e one may choose an open neighbourhood W of e with $W^{-1} = W \subseteq W^2 \subseteq U$. One may now follow verbatim the Kakutani argument. In broad outline: take a sequence of neighbourhoods $U_1, U_{1/2}, U_{1/4}, \dots$ of e that is a basis at e with

$$U_{1/2^n}^{-1} = U_{1/2^n} \subseteq U_{1/2^n}^2 \subseteq U_{1/2^{n-1}}, \text{ for } n \in \mathbb{N}.$$

Then construct a (Urysohn) function $f : G \rightarrow \mathbb{R}_+$ such that $f(x) \leq 2^{-n}$ iff $x \in U_{1/2^n}$ and define a metric by $d(g, h) = \sup_x |f(gx) - f(hx)|$. This is a right-invariant metric on G compatible with the topology, as Kakutani shows. \square

2.2 Analyticity

Notation. Write I for $\mathbb{N}^{\mathbb{N}}$ endowed with the product topology (treating \mathbb{N} as discrete) and $i|n := (i_1, \dots, i_n)$ for $i \in I$. In Section 2.3 we will consider the more general product space $J = \kappa^{\mathbb{N}}$, for κ an infinite cardinal (treated again as a discrete space). Although our concern here is with metric spaces, there are advantages in discussing analyticity in the broader context of topological spaces, arising from explicitly exposing their underlying topological nature. For X a Hausdorff space denote by $\mathcal{K} = \mathcal{K}(X)$ the family of compact subsets of a space X , $\mathcal{G} = \mathcal{G}(X)$ the open sets, and $\mathcal{F} = \mathcal{F}(X)$ the closed sets. If X is a metric space, X^* denotes its completion.

Definitions 1. A subset S of a topological space X is called *Souslin- $\mathcal{F}(X)$* if there is a determining system $\langle F(i|n) \rangle$ of closed sets giving S the representation

$$\bigcup_{i \in I} F(i), \text{ where } F(i) := \bigcap_{n \in \mathbb{N}} F(i|n). \quad (1)$$

2. A metric space S is said to be *analytic* (or, absolutely analytic) if it is Souslin- $\mathcal{F}(S^*)$, i.e. is Souslin in its completion.

We now introduce the \mathcal{K} -analytic sets of X ; when defined by reference to I these are Lindelöf and so for X metric are separable. Later, in Section 2.3, we consider a natural generalization to the non-separable metric context, and then the corresponding sets will no longer be Lindelöf.

Definitions (\mathcal{K} -analytic spaces). Adapting the notation of [Jay-Rog], for X and P Hausdorff spaces a map $K : P \rightarrow \wp(X)$ is called *compact-valued* if $K(p)$ is compact for each $p \in P$, and *singleton-valued* if each $K(p)$ is a singleton. K is *upper-semicontinuous* if, for each $p \in P$ and each open U in X with $K(p) \subseteq U$, there is a neighbourhood N of p such that $K(p') \subseteq U$ for each p' in N .

For the most part we take $P = I$, so that the neighbourhood N above may take the form $I(i|n) := \{i' \in I : i'|n = i|n = (i_1, \dots, i_n)\}$. The choice $P = J$ will be of interest in Section 2.3.

A subset in X is \mathcal{K} -analytic (relative to I) if it may be represented in the form $K(I)$ for some compact-valued, upper-semicontinuous map $K : I \rightarrow \wp(X)$. We say that X is a \mathcal{K} -analytic space if X itself is a \mathcal{K} -analytic set. A separable metric space S which is \mathcal{K} -analytic space will be termed *classically analytic*, or just *analytic*. It may be characterized by reference to its completion $X = S^*$ as being a Souslin- $\mathcal{F}(X)$ set, i.e. having a representation as in (1), where the determining system $\langle F(i|n) \rangle$ consist of closed sets in X qualified by the condition

$$\text{diam}_X F(i|n) < 2^{-n}.$$

As it is this last feature which makes Souslin representations useful, in the non-metric approach the singleton (or empty) sets $F(i)$ is replaced by compact sets (an idea due to Choquet, and studied also by Frolík, Sion and others – see [Jay-Rog]). Theorem AC below is an illustration.

We defer considering the (non-classical) context of non-separable metric spaces to Section 2.3. To achieve \mathcal{K} -analyticity there, generalizations of separability are needed: \mathbb{N} is replaced by an infinite cardinal κ in order to work relative to $J = \kappa^{\mathbb{N}}$, and countable sets by countable unions of discrete sets.

Notation (continued). We put $I(i|n) := \{i' : i'|n = i|n\}$ and $K(i|n) := K(I(i|n)) = \bigcup \{K(i') : i'|n = i|n\}$. The final result of this section is implicit in a number of situations, and goes back to Frolík's characterization of Čech-complete spaces as \mathcal{G}_δ in some compactification ([Frol-60]; see [Eng] §3.9); it provides a tool to lift theorems about Polish spaces to results on analytic metric spaces and as its title indicates may be seen as a generalized Cantor Theorem. We include a proof, for the sake of completeness, as it is short and required later.

Theorem AC (Analytic Cantor Theorem, cf. [Ost-AH]). *Let X be a Hausdorff space and $A = K(I)$ be \mathcal{K} -analytic in X , with K compact-valued and upper-semicontinuous.*

If F_n is a decreasing sequence of (non-empty) closed sets in X such that $F_n \cap K(I(i_1, \dots, i_n)) \neq \emptyset$, for some $i = (i_1, \dots) \in I$ and each n , then $K(i) \cap \bigcap_n F_n \neq \emptyset$.

Proof. If not, then $\bigcap_n K(i) \cap F_n = \emptyset$ and so, by compactness, $K(i) \cap F_p = \emptyset$ for some p , i.e. $K(i) \subseteq X \setminus F_p$. So by semicontinuity $F_p \cap K(I(i_1, \dots, i_n)) = \emptyset$ for some $n \geq p$, yielding the contradiction $F_n \cap K(I(i_1, \dots, i_n)) = \emptyset$. \square

The following immediate corollary (which refers to the fact that $B(x, r) = B(e_T, r)x$) is particularly useful in Section 3.

Proposition (Convergence criterion). *In a normed group, for $r_n \searrow 0$ and $\alpha_n = a_n \cdot \dots \cdot a_1$ with $\text{cl}B_{r_{n+1}}(a_{n+1}) \subseteq B_{r_n}(e)a_n$, if $X = K(I)$ is an analytic subset and $K(i_1, \dots, i_n) \cap B_{r_n}(\alpha_n) \neq \emptyset$ for some $i \in I$ and all n , then the sequence $\{\alpha_n\}$ is convergent.*

Proof. Indeed, $\alpha_n \rightarrow \alpha$, where $\{\alpha\} = K(i) \cap \bigcap_n F_n$ for $F_n = \text{cl}(B_{r_n}(\alpha_n))$ \square .

2.3 Non-separable analyticity

We have just termed a metric space S *analytic* (or, absolutely analytic) if it is Souslin- $\mathcal{F}(S^*)$, i.e. is Souslin in its completion. In a non-separable complete metric space X it is no longer possible to represent a Souslin- $\mathcal{F}(X)$ subset as a \mathcal{K} -analytic set relative to $I = \mathbb{N}^{\mathbb{N}}$. Various aspects of countability need generalization, as first realized by A. H. Stone, (see [Rog, Part 5]).

An (indexed) family of sets $\mathcal{H} := \{H_t : t \in T\}$ is *discrete* if every point has a neighbourhood meeting at most one set H_t (i.e. meets H_t for at most one index t). The family \mathcal{H} is σ -discrete if $\mathcal{H} = \bigcup_n \mathcal{H}_n$ where each indexed family $\mathcal{H}_n = \{H_t^n : t \in T_n\}$ is discrete.

Denoting by $wt(X)$, the weight of the space X (i.e. the smallest cardinality of a base for the topology), replacing $I = \mathbb{N}^{\mathbb{N}}$ by $J = \kappa^{\mathbb{N}}$ for $\kappa = wt(X)$, consider sets S with the following *extended Souslin* representation (following the terminology of [HJR])

$$\bigcup_{j \in J} F(j), \text{ where } F(j) := \bigcap_{n \in \mathbb{N}} F(j|n),$$

where the determining system $\langle F(j|n) \rangle$ consist of closed sets satisfying two conditions:

- (i) $\{F(j|n) : j|n \in \kappa^n\}$ is σ -discrete for each n ,
- (ii) $\text{diam}_X F(j|n) < 2^{-n}$, so that $F(j)$ is empty or single-valued, and so compact.

Then, by a theorem of Hansell [Han], the Souslin- $\mathcal{F}(X)$ sets are characterized as those having just such an extended Souslin representation (with $\kappa = wt(X)$). That is, they are *\mathcal{K} -analytic relative to J* . For other equivalent qualifications see [Han]; for a proper development of sets and spaces *\mathcal{K} -analytic relative to J* see [HJR] (especially for the fact that, in a topological space context, they are subparacompact rather than Lindelöf). By

Hansell's characterization theorem and Nikodym's theorem as it applies to the non-separable case (for which see [Ost-AB]), again as in Section 2.1, sets with an extended Souslin representation (qualified as above) have the Baire property. Finally, since $F : J \rightarrow \mathcal{K}(X)$ above is upper-semicontinuous, the Analytic Cantor Theorem continues to hold. Here the mapping F has properties additional to upper-semicontinuity (for which see [HJR]), related to the notion of σ -discreteness.

3 Analytic Shift Theorem

Theorem 3 is the main result of this section, a shift-compactness property of a group G relative to the translation action $x \rightarrow tx$, or more generally to trans-conjugacies $x \rightarrow ta^{-1}xa$; it assumes analyticity of a non-meagre set instead of completeness, so it is important to assess such an assumption in a normed group context. This turns out to imply *almost completeness* of the space, i.e. the existence of a dense completely metrizable subspace (a dense absolutely- \mathcal{G}_δ subspace). A non-meagre classically analytic set A in a metric space necessarily contains a non-meagre metrically complete subset ([Kur-1] IV.2 p. 88, since a classically analytic set is Baire in the restricted sense – Cor. 1 p. 482). So Theorem 1 below is a partial converse, which yields in Theorem 2 a characterization for normed groups. The proof given here is new, and is specific to the normed-group context (compare also Oxtoby's Theorem, e.g. in [Kech, Th. 8.17(i)]); it is based on the Analytic Cantor Theorem via the Convergence Criterion (of Section 2.2).

Theorem 1 (Analytic Baire Theorem). *In a normed group X under d_R^X , if X contains a non-meagre classically analytic set, then X is Baire and in fact, up to a meagre set, X is analytic (and separable).*

Proof. Let A be non-meagre and analytic with upper semi-continuous representation $A = K(I)$. Let $\{F_n^0 : n \in \omega\}$ be closed nowhere-dense subsets of X and G an arbitrary non-empty open set. Say $g \in G$. As A is analytic, by LSN (or more generally Nikodym's Theorem), A has the Baire property, so $A = U \setminus N \cup M$ for some open U and meagre M, N . As A is non-meagre, U is non-meagre and so non-empty. Say $u \in U \setminus N$. Since each mapping $\rho_t(x) = xt$ is a homeomorphism, $H := Gg^{-1}u$ is open and meets U in $u \in A$. As G is arbitrary, it suffices to show that H meets $X \setminus \bigcup_{m \in \omega} F_m^0$ in a point of A . (Here we are using the fact that a normed group which is locally Baire at

some point is Baire – for a more general result along these lines, see [THJ], Prop. 2.2.3.)

We choose inductively integers i_n , points x_n radii $r_n > 0$ with $r_{n+1} < r_n/2$ and nowhere dense closed sets $\{F_m^n : m \in \omega\}$ such that $K(i_1, \dots, i_n) \cap B(x_n, r_n)$ is non-meagre with $B(x_n, r_n) \subseteq B(x_{n-1}, r_{n-1}) \subseteq H$, and

$$K(i_1, \dots, i_n) \supseteq B(x_n, r_n) \setminus \bigcup_{m \in \omega} F_m^n \text{ and } B(x_n, r_n) \cap \bigcup_{k, m < n} F_m^k = \emptyset.$$

Begin by taking $x_0 = e_X$ and selecting r_0 arbitrarily so that $B(x_0, r_0) \cap K(I)$ is non-meagre. To verify the inductive step, note that

$$K(i_1, \dots, i_n) \cap B(x_n, r_n) = \bigcup_{m \in \omega} K(i_1, \dots, i_n, m) \cap B(x_n, r_n),$$

so there is i_{n+1} such that $K(i_1, \dots, i_n, i_{n+1}) \cap B(x_n, r_n)$ is non-meagre. So (again by Nikodym's Theorem) for some non-meagre open V , closed nowhere dense sets F_m^{n+1} and meagre N_{n+1} ,

$$K(i_1, \dots, i_n, i_{n+1}) \cap B(x_n, r_n) \setminus N_{n+1} = V \setminus \bigcup_{m \in \omega} F_m^{n+1}.$$

By analyticity, V is hereditarily separable, so we may choose x_{n+1}, r_{n+1} (with $r_{n+1} < r_n/2$) such that $x_{n+1} \in \text{cl}(B(x_{n+1}, r_{n+1})) \subseteq V \cap B(x_n, r_n) \setminus \bigcup_{m < n+1} F_m^k$ and

$$K(i_1, \dots, i_{n+1}) \cap B(x_{n+1}, r_{n+1})$$

is non-meagre. With the induction verified, by the the preceding Proposition (or from Th. AC), there is a such that $\{a\} = K(i) \cap \bigcap_{n \in \omega} B(x_n, r_n) \subseteq H$. So $a \in H$ (and $a = \lim_n x_n$). Furthermore, for any n we have $B(x_{n+1}, r_{n+1}) \cap F_n^0 = \emptyset$, so $a \notin \bigcup_{m \in \omega} F_m^0$.

As for the second claim, recall that $A = U \setminus N \cup M$ with U non-empty and N, M meagre with $N \subseteq U$ and $U \cap M = \emptyset$. Suppose that $B := B_\varepsilon(a) \subseteq U$, then $B \cap A = B \setminus N$ and so $B \setminus N$ is analytic. Now $\{Bx : x \in\}$ is an open cover of X so has a σ -discrete refinement, say $\mathcal{V} := \bigcup_{n \in \omega} \mathcal{V}_n$ with $\mathcal{V}_n := \{V_{tn} : t \in T_n\}$ discrete. Suppose that $V_{tn} \subseteq Bx_{tn}$. Put $N_{tn} := (N \cap B)x_{tn}$, which is meagre (since right-shifts are homeomorphisms); then $V_{tn} \setminus N_{tn} \subseteq (B \setminus N)x_{tn} \subseteq Ax_{tn}$. W.l.o.g. N_{tn} is a meagre \mathcal{F}_σ subset of V_{tn} and so $V_{tn} \setminus N_{tn} = (V_{tn} \cap (Ax_{tn})) \setminus N_{tn}$ is analytic and non-meagre. By Banach's Category Theorem ([Oxt, Ch. 16], also known as Banach's localization principle [Kel, Th. 6.35], or see [Jay-Rog, p. 42]), $N' := \bigcup_{n \in \omega} \bigcup \{N_{tn} : t \in T_n\}$ is meagre, since \mathcal{V}_n is discrete. By a theorem of Montgomery ([Mont]), for

each n the set $\bigcup\{V_{tn}\setminus N_{tn} : t \in T_n\}$ is analytic, being locally analytic on $\bigcup\{V_{tn} : t \in T_n\}$, and is non-meagre. As \mathcal{V} covers X , we have

$$X \setminus N' = \bigcup_{n \in \omega} \bigcup\{V_{tn}\setminus N_{tn} : t \in T_n\},$$

and so $X \setminus N'$ is analytic, being a countable union of analytic sets, and non-meagre. \square

In view of the observations in Section 2.3, and because Banach's Category Theorem allows category considerations to be applied to σ -discrete decompositions of a set, in much the same way, as they were applied to countable decomposition, we may re-assert Theorem 1 in a non-classical (non-separable) sense, that is:

Theorem 1'. *In a normed group X under d_R^X , if X contains a non-meagre analytic set, then X is Baire.*

As a corollary we obtain the following.

Theorem 2 (Characterization Theorem for Almost completeness). *In a separable normed group X under d_R^X , the following are equivalent:*

- (i) X is a non-meagre absolute \mathcal{G}_δ modulo a meagre set (i.e. is almost complete);
- (ii) X contains a non-meagre analytic subset;
- (iii) X is non-meagre analytic modulo a meagre set.

Proof. Clearly (i) implies (ii), since absolute \mathcal{G}_δ sets are analytic. We have just shown that (ii) implies (iii). Suppose that (iii) holds and that X equals a non-meagre analytic set A modulo a meagre set. Embed X in its completion X^* . Working in X^* , we may write $A = U \setminus N \cup M$ with M, N meagre, by Nikodym's Theorem. Since A is non-meagre, U is non-meagre. Replacing N by a larger meagre \mathcal{F}_σ set H , we see that A contains the non-meagre \mathcal{G}_δ subset $U \setminus H$. But $U \setminus H$ is an absolute \mathcal{G}_δ . But X equals $U \setminus H$ modulo a meagre set, and so (i) holds. \square

We quote the next result from [BOst-N] (Th. 5.5.B) in a more convenient form and give its proof, since it is short.

Displacements Lemma – Baire Case. *Working under d_R^X , for X a Baire space, A Baire non-meagre and for almost all $a \in A$, there is $\varepsilon = \varepsilon_A(a) > 0$ such that $A \cap Aa^{-1}xa$ is non-meagre for all x with $\|x\| < \varepsilon_A(a)$.*

Indeed, for any x with $\|x\| < \varepsilon_A(a)$, modulo meagre sets $Aa^{-1} \cap Aa^{-1}x \supseteq B(x, s(x))$ for some $s(x) > 0$, and so $A \cap Aa^{-1}xa \supseteq B(xa, s(x))$.

Proof. Omitting a meagre subset of A we may assume without loss of generality that $e \in Aa^{-1} = U \setminus N$ for some U open and N meagre; so as Aa^{-1} is non-meagre, there is $\varepsilon > 0$ such that $B(e, \varepsilon) \subseteq Aa^{-1}$ modulo meagre sets. For any x with $\|x\| < \varepsilon$ and $s := \min\{\varepsilon, \frac{1}{2}(\varepsilon - \|x\|)\} > 0$, we have $B(e, \varepsilon) \supseteq B(x, s) = B(e, s)x$. Indeed if $d_R(y, x) < s$, then $d(y, e) < d(y, x) + d(x, e) < s + \varepsilon = \frac{1}{2}(\varepsilon - \|x\|) + \|x\| < \varepsilon$. Modulo meagre sets: as $Aa^{-1}x \supseteq B(e, s)x$, we have

$$Aa^{-1} \supseteq Aa^{-1} \cap Aa^{-1}x \supseteq B(e, \varepsilon) \cap B(e, s)x \supseteq B(x, s),$$

which is non-meagre as the space X is Baire. \square

To interpret the next result, note that for z_n null, and a_m converging to t under d_R , we also have $ta_m^{-1}z_m a_m$ converging to t under d_R . Indeed,

$$\begin{aligned} d_R(ta_m^{-1}z_m a_m, t) &= \|ta_m^{-1}z_m a_m t^{-1}\| \leq \|ta_m^{-1}\| + \|z_m\| + \|a_m t^{-1}\| \\ &= \|a_m t^{-1}\| + \|z_m\| + \|a_m t^{-1}\| \rightarrow 0. \end{aligned}$$

The next theorem sharpens a corresponding result in [BOst-N], for a topologically complete context, by redeveloping a real-line argument in [BOst-KCC] under weaker hypotheses.

Theorem 3 (Analytic Shift Theorem). *In a normed group under the topology d_R^X , for $z_n \rightarrow e_X$ null, A a \mathcal{K} -analytic and non-meagre subset: for a non-meagre set of $a \in A$ with co-meagre Baire envelope, there is an infinite set \mathbb{M}_a and points $a_n \in A$ converging to a such that*

$$\{aa_m^{-1}z_m a_m : m \in \mathbb{M}_a\} \subseteq A.$$

In particular, if the normed group is topological, for quasi all $a \in A$, there is an infinite set \mathbb{M}_a such that $\{az_m : m \in \mathbb{M}_a\} \subseteq A$.

Proof. By the Analytic Baire Theorem (Th. 1 above), we may assume that all non-empty open sets are Baire and so non-meagre. Next note that

any non-meagre Baire set is equal modulo meagre sets to a non-meagre \mathcal{G}_δ . So if the Baire envelope of $t \in A$ with the asserted property has non-meagre complement, we may assume that this complement is analytic. It thus suffices to prove that a non-meagre analytic set A contains at least one point t for which there exists an infinite set \mathbb{M}_t and points $a_n \in A$ converging to t such that $\{ta_m^{-1}z_m a_m : m \in \mathbb{M}_t\} \subseteq A$.

We proceed to prove this. So write $A = K(I)$ with K upper-semicontinuous and single-valued. For each $n \in \omega$ we find inductively integers i_n , points x_n, y_n, a_n with $a_n \in A$, numbers $r_n > 0, s_n > 0$, analytic subsets A_n of A , and closed nowhere dense sets $\{F_m^n : m \in \omega\}$ such that

$$K(i_1, \dots, i_n) \supseteq A_n \cap (A_n a_n^{-1} x_n a_n) \supseteq B(y_n, s_n) \setminus \bigcup_{m \in \omega} F_m^n$$

and

$$y_n \in B(x_n a_n, r_n) \text{ and } B(y_n, s_n) \cap \bigcup_{m, k < n} F_m^k = \emptyset.$$

Assuming this done for n , since $K(i_1, \dots, i_n) = \bigcup_k K(i_1, \dots, i_n, k)$ is non-meagre, there is i_{n+1} such that $K(i_1, \dots, i_n, i_{n+1}) \cap A_n \cap (A_n a_n^{-1} x_n a_n)$ is non-meagre. Put $A_{n+1} := K(i_1, \dots, i_n, i_{n+1}) \cap A_n \cap (A_n a_n^{-1} x_n a_n) \subseteq K(i_1, \dots, i_{n+1})$. As A_n is non-meagre, we may pick $a_{n+1} \in A_{n+1}$ as in the Displacements Lemma, and $m(n)$ so large that $\|z_m\| < \varepsilon(a_{n+1}, A_{n+1})$ for $m \geq m(n)$. Pick $x_{n+1} = z_{m(n)}^{-1}$. Then, since $A_{n+1} \subseteq K(i_1, \dots, i_{n+1})$ and in view of the Displacements Lemma, there is r_{n+1} and closed nowhere dense sets $\{F_m^{n+1} : m \in \omega\}$ such that

$$K(i_1, \dots, i_{n+1}) \supseteq A_{n+1} \cap A_{n+1} a_{n+1}^{-1} x_{n+1} a_{n+1} \supseteq B(x_{n+1} a_{n+1}, r_{n+1}) \setminus \bigcup_{m \in \omega} F_m^{n+1}.$$

Since the set $\bigcup_{m, k < n+1} F_m^k$ is closed and nowhere dense, there is $y_{n+1} \in B(x_{n+1} a_{n+1}, r_{n+1})$ and $s_{n+1} > 0$ so small that $B(y_{n+1}, s_{n+1}) \subseteq B(x_{n+1} a_{n+1}, r_{n+1})$ and $B(y_{n+1}, s_{n+1}) \cap \bigcup_{m, k < n+1} F_m^k = \emptyset$. Hence

$$B(x_{n+1} a_{n+1}, r_{n+1}) \setminus \bigcup_{m \in \omega} F_m^{n+1} \supseteq B(y_{n+1}, s_{n+1}) \setminus \bigcup_{m \in \omega} F_m^{n+1}.$$

By the Analytic Cantor Theorem (Th. AC, Section 2), there is t with

$$\{t\} = K(i) \cap \bigcap_n B(y_n, s_n) \subseteq \bigcap_n A_n \cap (A_n a_n^{-1} x_n a_n).$$

So $t \in A$. Fix n . One has $t \notin \bigcup_{m \in \omega} F_m^n$ (since $B(y_{m+1}, s_{m+1}) \cap \bigcup_{k < m} F_k^n = \emptyset$ for each m), and so

$$t \in B(y_n, s_n) \setminus \bigcup_{m \in \omega} F_m^n \subseteq A_n \cap (A_n a_n^{-1} x_n a_n) \subseteq K(i_1, \dots, i_n) \subseteq A.$$

As $t \in A_n a_n^{-1} x_n a_n$, one has $ta_n^{-1} x_n^{-1} a_n = ta_n^{-1} z_{m(n)} a_n \in A_n \subseteq A$. So $\{ta_n^{-1} z_{m(n)} a_n : n \in \omega\} \subseteq A$. Moreover, $d_R(x_n a_n, t) = d_R(x_n, ta_n^{-1}) \rightarrow 0$, so since $x_n \rightarrow e$, we have $ta_n^{-1} \rightarrow e$, i.e. $a_n \rightarrow_R t$. \square

The next two results sharpen results of [BOst-N], by assuming analyticity in lieu of topological completeness; but, given Theorem 2, have almost identical proofs. We need these sharper version for later applications. We now deduce the following variant form of the Pettis Theorem – see [BOst-N] for the classical literature, which include variants such as [Fol] with an early application of a similar, doubled ‘difference set’ and [Hen] with higher-order versions in connection with uniform boundedness. Indeed, the higher order versions are the basis of the definition in [RoSo] of groups with the *Steinhaus property*.

Theorem 4 (Analytic Squared Pettis Theorem, [BOst-N] Th. 5.8). *For X a normed group, if A is analytic and non-meagre under d_R^X , then e_X is an interior point of $(AA^{-1})^2$.*

Proof. Suppose not. Then we may select $z_n \in B_{1/n}(e) \setminus (AA^{-1})^2$. As $z_n \rightarrow e$, we apply the preceding theorem (Th. 2) to A , to find $t \in A$, \mathbb{M}_t infinite and $t_m \in A$ for $m \in \mathbb{M}_t$ such that $tt_m^{-1} z_m t_m \in A$ for all $m \in \mathbb{M}_t$. So for $m \in \mathbb{M}_t$ one has $z_m \in AA^{-1}AA^{-1} = (AA^{-1})^2$, a contradiction. \square

We may now deduce:

Theorem 5 (Baire Homomorphism Theorem, cf. [Jay-Rog] §2.10, [BOst-N] Th. 11.11). *Let X and Y be normed groups with Y \mathcal{K} -analytic and X analytic and non-meagre under the respective right norm topologies. If $f : X \rightarrow Y$ is a Baire homomorphism, then f is continuous.*

Proof. For $f : X \rightarrow Y$ the given homomorphism, it is enough to prove continuity at e_X , i.e. that for any $\varepsilon > 0$ there is $\delta > 0$ such that $B_\delta(e_X) \subseteq f^{-1}[B_\varepsilon(e_Y)]$. So let $\varepsilon > 0$. We work with the right norm topology.

Being \mathcal{K} -analytic, Y is Lindelöf (cf. [Jay-Rog], Th. 2.7.1, p. 36) and metric, so separable; so choose a countable dense set $\{y_n\}$ and select $a_n \in f^{-1}(y_n)$. Put $T := f^{-1}[B_{\varepsilon/4}(e_Y)]$. Since f is a homomorphism, $f(Ta_n) = f(T)f(a_n) = B_{\varepsilon/4}(e_Y)y_n$. Note also that $f(T^{-1}) = f(T)^{-1}$, so $TT^{-1} = f^{-1}[B_{\varepsilon/4}(e_Y)]f^{-1}[B_{\varepsilon/4}(e_Y)^{-1}] = f^{-1}[B_{\varepsilon/4}(e_Y)^2] \subseteq f^{-1}[B_{\varepsilon/2}(e_Y)]$, by the triangle inequality.

Now $Y = \bigcup_n B_\varepsilon(e_Y)y_n$, so $X = f^{-1}(Y) = \bigcup_n Ta_n$. But X is non-meagre, so for some n the set Ta_n is non-meagre, and so too is T (as right-shifts are homeomorphisms). By assumption f is Baire. Thus T is Baire and non-meagre in X . Since X is analytic, T contains an analytic non-meagre subset. (If $T = (U \setminus M) \cup N$ with U open and M, N meagre, then U is non-meagre analytic and so is $U \setminus H$ for some meagre \mathcal{F}_σ set H with $H \supseteq M$ – cf. the opening remark of the section.) By the Analytic Squared Pettis Theorem (Th. 3), $(TT^{-1})^2$ contains a ball $B_\delta(e_X)$. Thus we have

$$B_\delta(e_X) \subseteq (TT^{-1})^2 \subseteq f^{-1}[B_{\varepsilon/4}(e_Y)^4] = f^{-1}[B_\varepsilon(e_Y)].$$

So f is continuous at the identity, so being a homomorphism is continuous. \square

4 Locally compact normed groups

Under either-sided norm topology, a normed group may be regarded a ‘unilaterally topological’ group and so is referred to as a right, or left, topological group. Since the development of Haar measure is also unilateral, it is natural to enquire whether there is a corresponding unilateral measure theory when the topology is locally compact. The answer is affirmative for compact right topological groups: the existence of a Haar measure was established relatively recently – see the papers by Paul Milnes and John Pym, e.g. [MP]. In the metrizable case of a normed group, if the group-norm has the Heine-Borel property, i.e. is *proper* as defined below, the answer is also affirmative; this follows from results in the literature of uniform structures, e.g. [Itz] or [Chr] – for a text-book account, see [SeKu] §7.4. For our context, a direct proof may be given, which appeals to an early result of Banach – see the extended web version of this paper.

Definitions. 1. Recalling that a set is *precompact* (or, relatively compact [Dug] XI.6) if its closure is compact, a metric d^X on X is *proper* if all the closed balls $\bar{B}_r^d(x) := \{y : d(x, y) \leq r\}$ are compact, i.e. the metric has the *Heine-Borel* property: closed and bounded is equivalent to compact. (In geodesic geometry a proper metric space is called ‘finitely compact’, since an infinite bounded set has a point of accumulation – see [Bus2], or [BH] for a more recent text-book account of the extensive use of this concept.)

2. Say that the group-norm $\|\cdot\|$ on X is *right* (resp. *left*) *proper* if d_R^X (resp. d_L^X) is a proper metric, i.e. norm-bounded sets are *precompact*, equivalently closed balls are compact.
3. Say that a group-norm $\|\cdot\|$ is *proper* if it is either right-proper or left-proper.

Note that a Hausdorff space has a proper metric iff it is locally compact and second countable (a result due to H.E. Vaughan, for which see [Bus1] Th.1.21, where the metrization in the non-compact case is derived from a metrization of a one-point compactification; cf. [Ro-B, Th. 3.10], [Nie]). Compare also [SeKu] §7.3.

Theorem H (Existence of Haar measure for normed groups). *In a proper normed group X under its right norm topology (so having the Heine-Borel property), there exists a right-invariant Haar measure on the Borel sets of X .*

Our main corollary of Th. H is again a shift-theorem; we demonstrate below some of the measure-theory counterparts in the normed-group setting. We need the following lemma.

Displacements Lemma – measure case (cf. [WKh], [Kem] Th. 2.1 in \mathbb{R}^d with $B_i = E$, $a_i = t$; cf. [BOst-N], Th. 5.5M). *In a normed group that is locally compact under its right normed topology equipped with a right-invariant Haar measure μ , if E is non-null Borel, then $f(x) := \mu[E \cap (E+x)]$ is continuous at $x = e_X$, and so for some $\varepsilon = \varepsilon(E) > 0$*

$$E \cap (Ex) \text{ is non-null, for } \|x\| < \varepsilon.$$

Proof. The proof of Theorem 61.A of [Hal] (Ch. XII, p. 266), which is unilateral, demonstrates that $f(x)$ is continuous. \square

The very strong assumption of local compactness in the next theorem, obviates the need to appeal to analyticity as in Theorem 2, since for the non-null set T below to contain a non-null compact subset it is enough to assume measurability, rather than the stronger analyticity. So the argument of Section 3 is now simpler here; the much simpler real line case goes back to Kestelman [Kes], reproved later in the measure case quite simply by Borwein and Ditor [BoDi].

Theorem 6 (Kestelman-Borwein-Ditor Theorem – normed Haar version). *In a normed group X whose right norm topology has the Heine-Borel property, if $\{z_n\} \rightarrow e_X$ (a null sequence converging to the identity) and T is (right) Haar-measurable, then for almost all $t \in T$ there is an infinite set \mathbb{M}_t such that*

$$\{z_m t : m \in \mathbb{M}_t\} \subseteq T. \quad (\text{embed})$$

Proof. The conclusion of the theorem is inherited by supersets (is upward hereditary), so without loss of generality we may assume that T is compact and non-null.

We begin by showing that T contains at least one element t with the property (embed). Working inductively, we define non-null compact subsets of T (of possible translators) B_m of diameter less than 2^{-m} , as follows. With $n = 0$, we take $B_0 = T$. Given B_n compact in T , choose N such that $\|z_k\| < \min\{\frac{1}{2}\|x_n\|, \varepsilon(B_n)\}$, for all $k > N$. Let $x_{n+1} = z_N \in Z$; then by the Displacement Lemma $B_n \cap (B_n x_n^{-1})$ is non-null, and compact (as right shifts are homeomorphisms). We may now choose a non-null compact subset B_{n+1} of T with diameter less than 2^{-n-1} such that $B_{n+1} \subset B_n \cap (B_n x_n^{-1}) \subseteq B_n$. By compactness, the intersection $\bigcap_{n \in \mathbb{N}} B_n$ is non-empty. Let $t \in \bigcap_{n \in \mathbb{N}} B_n \subset T$. Now $tx_n \in B_n \subset T$, as $t \in B_{n+1}$, for each n . Hence $\mathbb{M}_t := \{m : z_m = x_n \text{ for some } n \in \mathbb{N}\}$ is infinite and $\{tz_m : m \in \mathbb{M}_t\} \subseteq T$.

We have shown that the Borel subset $T^* := T \cap \bigcap_{n \in \mathbb{N}} \bigcup_{k \in \mathbb{N}} T z_n^{-1}$ of T is non-empty. Suppose that the Borel subset $S := T^* \setminus T$ of T is non-null. Then there is an element $s \in S$ and an infinite set \mathbb{M}_s such that $\{sz_m : m \in \mathbb{M}_s\} \subseteq S \subseteq T$, so $s \in T$ a contradiction since $s \in T^* \setminus T$. So S is null. \square

For normed groups, as an immediate corollary, we find that the measure case of the Pettis Theorem retains its usual simple form, in contrast to the squared form needed for the Baire case (Th. 4 above).

Theorem 7 (Pettis-Steinhaus Theorem – normed Haar version, cf. [BOst-N] Th. 6.5). *In a normed group X whose norm topology has the Heine-Borel property, for S Borel in the right norm topology and of finite positive measure for the right-invariant Haar measure, the difference set SS^{-1} has e_X as interior point.*

Proof. We work in the right norm topology. Suppose the theorem false. Then for some S Borel and with positive right-Haar measure there is for each positive integer n a point $z_n \in B_{1/n}(e) \setminus (SS^{-1})$. Since $z_n \rightarrow e_X$, by the

Kestelman-Borwein-Ditor Theorem, for almost all $s \in S$ there is an infinite \mathbb{M}_s such that $\{z_m s : m \in \mathbb{M}_s\} \subseteq S$. Then for any $m \in \mathbb{M}_s$, $z_m s \in S$, i.e. $z_m \in SS^{-1}$, a contradiction. \square

As a further corollary we deduce the Continuity Theorem for measurable homomorphisms (between normed groups with the Heine-Borel property). As a particular case, we deduce that if inversion $x \rightarrow x^{-1}$ is measurable, then the normed group is a topological group.

Theorem 8 (Homomorphism Continuity Theorem – normed Haar version, cf. [BOst-N] Th. 11.11). *For a normed group X whose right norm topology has the Heine-Borel property and f a homomorphism from X to a normed group Y , if f is Haar-measurable, then f is continuous.*

Proof. We work with the right norm topologies without loss of generality, since inversion is a homomorphism and also an isometry from the left to the right norm topology ([BOst-N], Prop. 2.5). We claim that it is enough to prove the following: for any non-empty open G in X and any $\varepsilon > 0$ there is a non-empty open $V \subseteq G$ with $\text{diam}(f(V)) < \varepsilon$. Indeed the claim implies that for each $n \in \mathbb{N}$ the set $W_n := \bigcup \{V : \text{diam}(f(V)) < 1/n \text{ and } V \text{ is open and non-empty}\}$ is dense and open in X . Hence, as X is locally compact it is Baire, and so the intersection $\bigcap_{n \in \mathbb{N}} W_n$ is a non-empty set containing continuity points of f ; but f is a homomorphism, so is continuous everywhere.

Fix G non-empty and open and $\varepsilon > 0$. As f is measurable, by a theorem of Lusin ([Hal], end of Section 55) it is continuous when restricted to $X \setminus M$ for some measurable set M with $m(M) < \varepsilon$. Passing to a subset, there is a measurable set H of positive measure in X disjoint from M and with $B_\varepsilon^X(H) \subseteq G$ such that $\text{diam}_X(H) < \varepsilon/6$ and $\text{diam}_Y(f(H)) < \varepsilon/6$.

Note that $HH^{-1} \subseteq B_\varepsilon^X(e_X)$ (as $\|h'h^{-1}\| \leq \|h'\| + \|h\|$) and likewise $f(H)f(H)^{-1} \subseteq B_{\varepsilon/3}^Y(e_Y)$. By the Pettis Theorem (Th. 3 above), there is a non-empty open set U contained in HH^{-1} . Fix $h \in H$ and put $V := Uh$. Then

$$V = Uh \subseteq HH^{-1}h \subseteq B_\varepsilon(e_X)h = B_\varepsilon(h) \subseteq B_\varepsilon^X(H) \subseteq G,$$

and so $V \subseteq G$; moreover, since f is a homomorphism,

$$f(V) = f(Uh) = f(U)f(h) \subseteq f(H)f(H)^{-1}f(h) \subseteq B_{\varepsilon/3}^Y(f(h)),$$

and so $\text{diam}_Y f(V) < \varepsilon$, as claimed. \square

Theorem 9 (On measurable inversion, cf. [BOst-N], Th. 3.41). *For a normed group X whose right norm topology d_R^X has the Heine-Borel property, if the inversion $x \rightarrow x^{-1}$ regarded as a map from (X, d_R) to (X, d_R) is a (Haar) measurable function, then X is a topological group.*

Proof. To apply Th. 7 we need to interpret inversion as a homomorphism between normed groups. To this end, define $X^* = (X, *, d^*)$ to be the metric group with underlying set X with multiplication $x * y := yx$ and metric $d^*(x, y) = d_R(x, y)$. Then X^* is isometric with (X, \cdot, d_R) under the identity and d^* is left-invariant, since $d^*(x*y, x*z) = d_R(yx, zx) = d_R(y, z) = d^*(y, z)$. Thus X^* has the Heine-Borel property. Now $f : X \rightarrow X^*$ defined by $f(x) = x^{-1}$ is a homomorphism, which is measurable (by the isometry). Hence f is continuous and so is right-to-right continuous. Now by the Equivalence Theorem ([BOst-N] Th. 3.4), the normed group X is topological. \square

Using the same ideas, one may prove:

Theorem 10. *In the setting of Theorem 8, if each map $\gamma_g(x) := gxg^{-1}$ is measurable, then X is a topological group.*

Solecki and Srivastava [SolSri] prove a similar result in the context of a group which is a Baire space hat is separable, metrizable with continuous right-shifts $\rho_t(s) = st$ but with Baire-measurable left-shifts $\lambda_s(t) = st$.

We close by noting the following.

Theorem 11 (Uniqueness of Banach-Haar measure). *In a locally compact normed group with the Heine-Borel property, the left/right invariant Haar measure is unique up to proportionality.*

This may be deduced from Cartan's Approximation Theorem (cf. [Na] Ch. II.9, Th. 3 p. 115) provided we restrict attention to symmetric approximating functions, once one has checked the extent to which Cartan's original arguments may be presented unilaterally: details are in the extended website version of this paper.

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